Subscripts

b = bulk
fd = fully developed
i = inlet
q = wall heat flux, no internal generation
qS = insulated wall, internal heat generation
T = prescribed wall temperature, no internal generation
TS = zero wall temperature, internal heat generation

LITERATURE CITED

1. Topper, L., *Chem. Eng. Sci.*, 5, 13-19 (1956).
2. Sparrow, E. M., and R. Siegel, *Nucl. Sci. Eng.*, 4, 239-254 (1958).
3. Inman, R. M., *Internat. J. Heat and Mass Transfer*, 5, 1053-1059 (1962).
4. Siegel, R., and E. M. Sparrow, *J. Heat Transfer*, C81, 280-290 (1959).
5. Poppendiek, H. F., and L. D. Palmer, Oak Ridge National Laboratory, *Rept. ORNL-1701* (1954).
6. Loeffler, A. L., Jr., *Nucl. Sci. Eng.*, 2, 547 (1957).
7. Jakob, Max, "Heat Transfer," Vol. 1, Wiley, New York (1949).
8. Brown, G. M., *Am. Inst. Chem. Engrs. Journal*, 6, 179-183 (1960).
9. Novotny, J. L., Research in progress, Mechanical Engineering Department, University of Minnesota, Minneapolis, Minnesota.
10. Cess, R. D., and E. C. Shaffer, *Appl. Sci. Res.*, A8, 339-347 (1959).

Manuscript received March 15, 1963; revision received July 8, 1963; paper accepted July 10, 1963.

Behavior of Non-Newtonian Fluids in the Entry Region of a Pipe

MORTON COLLINS and W. R. SCHOWALTER

Princeton University, Princeton, New Jersey

The current trend toward a more scientific approach to engineering design has brought about the need for fundamental information concerning the behavior of non-Newtonian fluids in motion. Such information can be obtained by solving the differential equations of motion for non-Newtonian fluids. Exact solutions to these equations are possible only for relatively simple geometries and constitutive equations (the equations relating the stress condition of a fluid to its velocity field). Many problems are inherent in obtaining solutions to the equation of motion. A review by Oldroyd (*10*) treats some of these problems.

Approximate solutions to the equation of motion have been of vital importance in the advancement of Newtonian fluid mechanics, and hence it seems likely that similar approximate solutions will be valuable in the study of non-Newtonian fluids in motion. Thus the use of boundary-layer theory for non-Newtonian systems has been discussed by Schowalter (*14*), and specific applications have been reported by Acrivos, et al. (*1*) and Collins and Schowalter (*5*). In the present paper principles of boundary-layer theory have been applied to determine the behavior of a non-Newtonian fluid in the entry region of a pipe. The results apply to laminar flow of viscous pseudoplastic fluids whose rheological behavior can be approximated by a power-law relation between shear stress and velocity gradient.

Earlier workers (*3, 15*) have used approximate forms of boundary-layer theory to estimate the entry length for axisymmetric pipe flow. The present analysis differs from these in that a more realistic treatment of the boundary layer is utilized. The results provide estimates of the pressure drop which occurs in the entry region, the length of pipe required to establish fully developed laminar flow, and the shape of velocity profiles in the entry region. The technique used is an extension of that previously described by the authors (*5*). It is believed that the results provide a better description of velocity profiles in the entry region than has previously been available. The results should be helpful to persons interpreting pressure loss and heat transfer data taken under conditions such that the shape of the velocity profile is affected by proximity of the pipe entrance.

Morton Collins is with Esso Research and Engineering Company, Linden, New Jersey.

THEORY

Development of the basic equations is analogous to the procedure previously employed for flow in a two-dimensional channel (*5*). Consequently only essential points will be treated here.

Consider a pipe of radius a and a cylindrical coordinate system with its origin at the entry. The x axis is on the pipe center line, and the r axis is normal to the center line. The fluid is assumed to enter the pipe at $x = 0$ with a flat velocity profile ($u = U_o$, $v = 0$). Near the entry boundary-layer techniques are applicable, while in the region approaching fully developed flow the velocity profile may be described in terms of perturbations to fully developed flow. Results from the two methods of solution are joined at an appropriate value of x.

The analysis applies to fluids which can be characterized by Bird's generalization of the power law, which assumes that any effect of the third invariant is of second order (*2*):

$$\mu_{\text{eff}} = K\left(\frac{1}{2} I_2\right)^{\frac{n-1}{2}} \tag{1}$$

For the axisymmetric flow under consideration the approximation

$$I_2 \cong 2\left(\frac{\partial u}{\partial r}\right)^2 \quad (1a)$$

is made. This permits one to write

$$\tau_{rx} \cong -K\left[\left(\frac{\partial u}{\partial r}\right)^2\right]^{\frac{n-1}{2}} \frac{\partial u}{\partial r} \quad (2)$$

Solution Near the Entry (Downstream Solution)

The boundary-layer equation for the x component of velocity can be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = U\frac{dU}{dx} + \frac{Kn}{\rho}\left[\left(\frac{\partial u}{\partial r}\right)^2\right]^{\frac{n-1}{2}}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{rn}\frac{\partial u}{\partial r}\right] \quad (3)$$

Equation (3) is valid in the thin boundary layer near the pipe wall where $u = u(x, r)$. Outside of the boundary layer $u \cong U(x)$.

To convert Equation (3) into a set of ordinary differential equations define a stream function

$$\psi = -a^2 U_o \sum_{i=1} \sigma^i f_i(\eta) \quad (4)$$

where

$$\eta = \frac{\left(1 - \frac{r^2}{a^2}\right)}{2\sigma}$$

$$\sigma = \left[\frac{2x}{aN_{Re}}\right]^{\frac{1}{n+1}}$$

Velocities are given in terms of ψ by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r}; \quad v = -\frac{1}{r}\frac{\partial\psi}{\partial x} \quad (5)$$

Then from Equations (4) and (5)

$$u = U_o \sum_{i=1} \sigma^{i-1} f_i'(\eta)$$

$$v = \frac{a^2 U_o}{r(n+1)x}\left\{\sum_{i=1} i\sigma^i f_i - \eta \sum_{i=1} \sigma^i f_i'\right\} \quad (6)$$

The velocity outside of the boundary layer can be expressed as a function of σ, which in turn is a function of x

$$U(x) = U_o\left[1 + \sum_{i=1} \sigma^i K_i\right] \quad (7)$$

where the constants K_i will be determined later. By combining Equations (3), (6), and (7) one arrives at a differential equation containing terms made up of coefficients which are functions of η alone multiplied by σ raised to different powers (0, 1, 2 ...). When coefficients of like powers of σ are equated, an infinite set of ordinary differential equations is obtained, the first one being

$$2^{n-1} n(n+1) f_1''' + f_1 f_1'' \left[(f_1'')^2\right]^{\frac{1-n}{2}} = 0 \quad (8)$$

with boundary conditions formed from the requirement of no slip at the wall and $u = U$ at $\eta = \infty$ (when only one side of the pipe center line is considered):

$$f_1(0) = f_1'(0) = 0$$
$$f_1'(\infty) = 1$$

Strictly speaking the boundary condition at infinity is inconsistent with the assumption of a uniform velocity near the center of a pipe of finite radius. However in the region where the boundary-layer equations are applied the effective boundary layer is sufficiently thin so that difficulties are avoided. Coefficients of higher powers of σ yield equations for f_2 through f_4. These equations are presented in the Appendix.*

In order to solve Equations (8) and (A1) through (A3) one must have numerical values for the constants $K_1, K_2, \ldots$ For large values of η

$$f_i = (K_{i-1})\,\eta + A_i; \; i = 1, 2, \ldots \quad (9)$$

where $K_o = 1$.

Continuity requires

$$2\pi\,\psi_{r=a} - 2\pi\,\psi_{r=o} = \pi a^2 U_o \quad (10)$$

Equations (4) and (10) yield

$$\psi_{r=o} = -\frac{a^2 U_o}{2}$$

or

$$\sum_{i=1} [\sigma^i f_i]_{\eta=1/2\sigma} = \sum_{i=1} \sigma^i\left(\frac{K_{i-1}}{2\sigma} + A_i\right) = \frac{1}{2} \quad (11)$$

Therefore

$$K_i = 2[(K_{i-1})\,\eta_1 - f_i(\eta_1)]; \; i = 1, 2, \ldots \quad (12)$$

where η_1 is some point outside of the boundary layer. Hence K_1 can be determined from the solution to Equation (8) and used to solve Equation (A1) from which K_2 can be determined, etc.

In principle one can form as many ordinary differential equations of the type Equations (8) and (A1) through (A3) as desired. The present analysis employs four equations for f_1 through f_4. Knowing f_1 through f_4 one can compute the velocity in the boundary layer by writing Equation (6) in the form

$$u \cong U_o \sum_{i=1}^{i=4} f_i' \sigma^{i-1} \quad (13)$$

Solution Approaching Fully Developed Flow (Upstream Solution)

In the region approaching a fully developed state one describes the flow in terms of a perturbation to the fully developed velocity profile. Since the perturbation grows as one moves upstream from the fully developed flow, the solution has been coined the *upstream solution*. An approximation to the equation of motion is obtained (see discussion section)

$$u\frac{\partial u}{\partial x} + \frac{v}{a}\frac{\partial u}{\partial R} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{Kn}{\rho a^{n+1}}\left[\left(\frac{\partial u}{\partial R}\right)^2\right]^{\frac{n-1}{2}}\left[\frac{\partial^2 u}{\partial R^2} + \frac{1}{Rn}\frac{\partial u}{\partial R}\right] \quad (14)$$

where $R = r/a$. Partial differentiation of Equation (14) with respect to R yields

* Appendix has been deposited as document 7702 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for \$1.25 for photoprints or for 35-mm. microfilm.

$$u\frac{\partial^2 u}{\partial x \partial R}+\frac{v}{a}\frac{\partial^2 u}{\partial R^2}-\frac{v}{aR}\frac{\partial u}{\partial R}=\frac{Kn}{\rho a^{n+1}}\frac{\partial}{\partial R}\left\{\left[\left(\frac{\partial u}{\partial R}\right)^2\right]^{\frac{n-1}{2}}\left[\frac{\partial^2 u}{\partial R^2}+\frac{1}{Rn}\frac{\partial u}{\partial R}\right]\right\} \tag{15}$$

where the assumption $p = p(x)$ has been used along with the continuity equation

$$a\frac{\partial uR}{\partial x}+\frac{\partial(vR)}{\partial R}=0 \tag{16}$$

A perturbation velocity u^* is defined by

$$u = U_o\left(\frac{1+3n}{1+n}\right)(1-R^{\frac{n+1}{n}})-u^* \tag{17}$$

Substituting this expression into Equation (15) one obtains

$$\left(\frac{1+3n}{1+n}\right)(1-R^{\frac{n+1}{n}})\frac{\partial^2 u^*}{\partial x \partial R}+\left(\frac{1+3n}{n^2}\right)(n-1)R^{\frac{1-2n}{n}}\int_R^1 R\frac{\partial u^*}{\partial x}dR=$$
$$\frac{KnU_o^{n-2}}{\rho a^{n+1}}\left(\frac{1+3n}{n}\right)^{n-1}\left\{R^{\frac{n-1}{n}}\frac{\partial^3 u^*}{\partial R^3}+\left(\frac{3n-2}{n}\right)R^{-\frac{1}{n}}\frac{\partial^2 u^*}{\partial R^2}-\left(\frac{2n-1}{n^2}\right)R^{-\left(\frac{1+n}{n}\right)}\frac{\partial u^*}{\partial R}\right\} \tag{18}$$

where terms involving the perturbation velocity u^* or its derivatives to powers greater than 1 have been neglected. Equation (18) can be solved by separation of variables. Set

$$u^* = U_o\sum_i c_i \exp(-\lambda_i X)\,\varphi_i'(R) \tag{19}$$

where

$$X=\frac{KU_o^{n-2}x(1+3n)^{n-2}(n+1)}{\rho n^{n-2}a^{n+1}}=2^{n-1}\left(\frac{1+3n}{n}\right)^{n-2}(n+1)\,\sigma^{n+1} \tag{20}$$

Substitution of Equation (19) into (18) results in a set of ordinary differential equations for the φ_i:

$$\lambda_i(1-R^{\frac{n+1}{n}})\varphi_i''+\left(\frac{n+1}{n^2}\right)(n-1)\lambda_i R^{\frac{1-2n}{n}}\int_R^1 R\varphi_i'\,dR+R^{\frac{n-1}{n}}\varphi_i''''+\left(\frac{3n-2}{n}\right)R^{-\frac{1}{n}}\varphi_i'''-\left(\frac{2n-1}{n^2}\right)R^{-\left(\frac{1+n}{n}\right)}\varphi_i''=0 \tag{21}$$

It is advantageous to transform the dependent variable in Equation (21) by setting

$$\theta_i' = R\varphi_i'$$

Then Equation (21) becomes

$$\theta_i''''-\frac{2}{nR}\theta_i'''+\left[\frac{2n+1}{n^2R^2}+R^{\frac{1}{n}}(R^{-1}-R^{\frac{}{n}})\lambda_i\right]\theta_i''-\left[\frac{2n+1}{n^2R^3}+R^{\frac{1}{n}}(R^{-2}-R^{\frac{1}{n}-1})\lambda_i\right]\theta_i'-\left(\frac{n+1}{n}\right)\left(\frac{n-1}{n}\right)R^{\frac{2}{n}-2}\lambda_i\theta=0 \tag{22}$$

Since the values of c_i in Equation (19) have not yet been fixed, one may assign any nonzero value to $\varphi_i'(0)$. It is convenient to set $\varphi_i'(0) = 1$. When this is done, one can show from conditions of symmetry and no slip at the wall that the boundary conditions are

$$\theta_i(0)=\theta_i'(0)=\theta_i(1)=\theta_i'(1)=0$$
$$\theta_i''(0)=1$$

Solution of Equation (22) results in a set of eigenfunctions θ_i and corresponding eigenvalues λ_i for which the boundary conditions are met. The correct values of c_i are those which will provide the best fit of the upstream solution to the downstream solution at the point where the two are joined. The values of c_i are found by minimizing the integral

$$\int_0^1 [U_o \Sigma\, c_i \exp(-\lambda_i X_j)\,\varphi_i'(R)-u_d^*]^2\,dR \tag{23}$$

where u_d^* is the value of the perturbation velocity evaluated from the downstream solution at the point where the solutions are joined.

PRESSURE LOSS

In the fully developed region of laminar flow in a pipe the change in pressure $|\Delta p|$ over a length Δx is given by

$$\frac{|\Delta p|}{\frac{1}{2}\rho U_o^2}=\frac{2^{n+2}\left(\frac{1+3n}{n}\right)^n}{N_{Re}}\frac{\Delta x}{a} \tag{24}$$

The pressure loss between $x = 0$ and some point x_1 in the fully developed region may be expressed by

$$\frac{|\Delta p|}{\frac{1}{2}\rho U_o^2}=\frac{2^{n+2}\left(\frac{1+3n}{n}\right)^n}{N_{Re}}\frac{x_1}{a}+C \tag{25}$$

where C is found from a momentum balance to be

$$C=2\left[\left(\frac{1+3n}{2n+1}\right)-1\right]-\frac{2^{n+1}}{U_o^n}\left[\int_o^{\sigma_D^{n+1}}\left[\left(\frac{\partial u}{\partial R}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial R}\,d\sigma^{n+1}\right]_{R=1}-\frac{2^{n+2}\left(\frac{1+3n}{n}\right)^n}{N_{Re}}\frac{x_D}{a} \tag{26}$$

The constant C represents the additional pressure loss occurring in the entry over and above that of fully developed flow.

It should be noted that all of the above equations consider the fluid to have a flat velocity profile $u = U_o$ at x

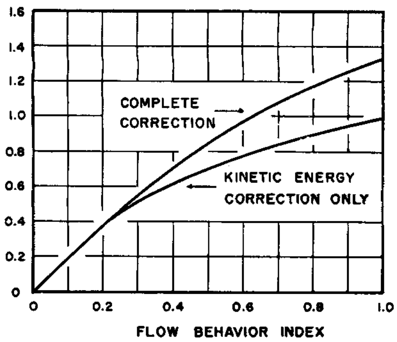

Fig. 1. Entry pressure drop correction as a function of flow behavior index.

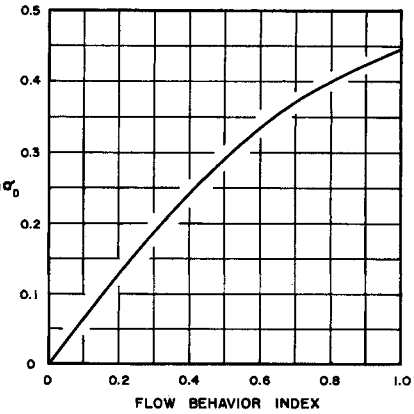

Fig. 2. Entry length as a function of flow behavior index.

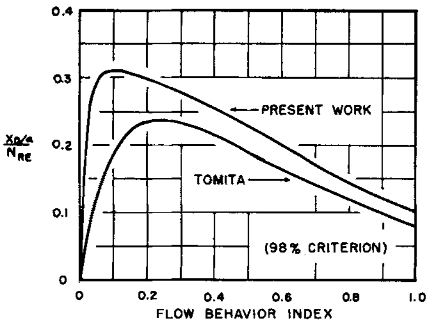

Fig. 3. Entry length as a function of flow behavior index.

= 0. Hence the change in pressure caused by acceleration of the fluid from a reservoir at $x < 0$ to the average channel velocity U_o at $x = 0$ has not been included.

NUMERICAL PROCEDURE

Equations (8), (*A*1) through (*A*3), and (22) were solved with the Gill modification of the Runge-Kutta technique (*12*). Methods used to find initial conditions, when necessary, and the eigenvalues λ_i in Equation (22) are described elsewhere (*4*).

The downstream solution [Equation (13)] was considered valid only as long as the absolute value of each successive term in Equation (13) decreased. At the point where this ceased to be true the downstream solution was joined to the upstream solution.

Equation (22) was solved for four values of n between 0.4 and 1.0. For each value of n three eigenvalues and the corresponding eigenfunctions were determined. For example for $n = 0.8$ one finds

$$\lambda_1 = 39.14$$

$$\lambda_2 = 117.95$$

$$\lambda_3 = 237.24$$

Thus three terms of Equation (19) were used in expressing the upstream solution. Minimization of the integral (23) at the point where the upstream and downstream solutions were joined for $n = 0.8$ ($\sigma_j = 0.0635$) results in

$$c_1 = 0.85$$

$$c_2 = -0.32$$

$$c_3 = 0.34$$

RESULTS

The results presented in Figures 1 through 7 are based upon calculations performed for $n = 0.4$, 0.6, 0.8, and 1.0. Figure 1 shows the dependence of the constant C in Equation (25) on the flow behavior index n. Also included in Figure 1 is a curve showing the dependence of C upon n if viscous dissipation is neglected, and the pressure loss in the entry is equated to the change in kinetic energy of the fluid between $x = 0$ and $x = x_D$.

The entry length x_D was selected as the distance from the entry at which the center-line velocity had reached 98% of its fully developed value. A smooth curve with no sudden changes in slope is obtained if one plots $n\sigma_D$ vs. flow behavior index, shown in Figure 2. The more conventional plot of $x_D/(aN_{Re})$ vs. n is presented in Figure 3.

Figures 4 through 6 present a comparison of velocity profiles for different values of flow behavior index at positions in the pipe which are equivalent.

DISCUSSION

The approximations of Equations (1*a*) and (2) are consistent with the boundary-layer approximations used in the downstream solution. Errors incurred by neglecting products of u^* and its derivatives in the development of Equation (22) are similar to those discussed in earlier work (*5*). Use of Equations (1*a*) and (2) in derivation of the upstream Equation (14) requires

$$\frac{\partial u^*}{\partial r} >> \frac{\partial v}{\partial x}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u^*}{\partial r}\right] >> \frac{\partial^2 u}{\partial x^2}$$

It is also assumed that these inequalities are not affected by the partial differentiation used to obtain Equation (15).

Validity of the upstream approximations depends upon the flow behavior index n, Reynolds number, and axial and radial position. The approximations are always good near the wall but are not always fulfilled near the center line and at isolated radial positions. It is also recognized that validity of the power law must break down as the center line is approached. Similar comments apply to the upstream solution given in (*5*).

Use of Equation (13) near the entry, and joining (13) to the upstream solution at the point where (13) begins to diverge, results in an excellent match of upstream and downstream solutions.

Figure 1 indicates that for values of $n < 0.3$ the entry correction is essentially the same as that calculated con-

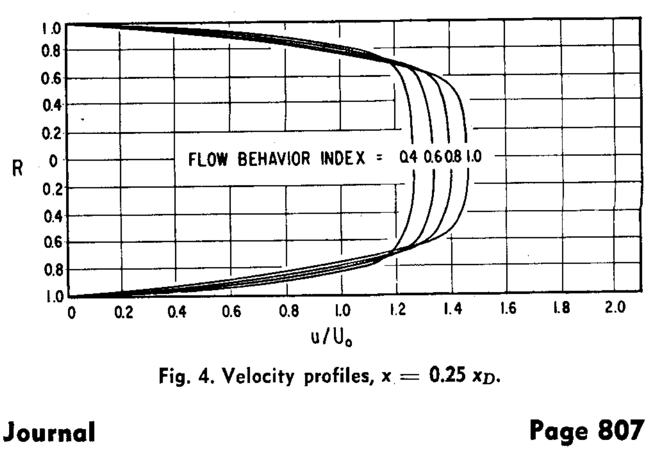

Fig. 4. Velocity profiles, $x = 0.25\ x_D$.

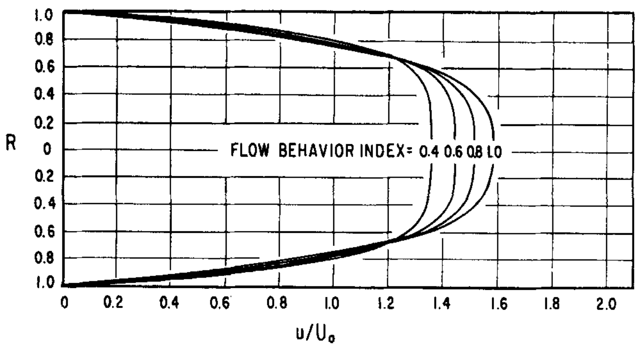

Fig. 5. Velocity profiles, $x = 0.5\ x_D$.

sidering only the kinetic energy increase of the fluid. The kinetic energy calculation is quite simple and hence should be used for low values of n.

It is seen from Figure 3 that if one considers the flow of fluids with different values of n to be on an equivalent basis when the Reynolds numbers are identical, the entry length (in terms of pipe radii) actually increases as the flow behavior index decreases from 1 to about 0.1.

No results have been calculated for $n < 0.4$ owing to extremely slow convergence in the calculation of eigenvalues for Equation (22). Entry length results for low values of n were obtained by interpolation between $n = 0.4$ and $n = 0$.

The axisymmetric entry problem has been previously considered by Goldstein (*6*), Langhaar (*8*), Nikuradse (*9*), and Schiller (*13*) for Newtonian fluids, and by Bogue (*3*) and Tomita (*15*) for non-Newtonian fluids. Table 1 compares these results with the results of the present analysis. Comparisons are made with 99% of the fully developed center-line velocity as the criterion determining entry length, as many other investigators have used this criterion. The 98% criterion previously mentioned is felt to be more useful in view of the large distance required for development of the center-line velocity from 98 to 99% of its fully developed value. Any errors incurred will be magnified to a greater extent with the 99% criterion.

Goldstein used methods somewhat analogous to those used in the present analysis with the exception that in his work the upstream equation was refined by solving a second time including some of the terms neglected in the initial solution. In his final result only one term in the upstream analysis was retained.

Langhaar's solution was obtained by linearizing the equation of motion and expressing the solution as a series of Bessel functions. The linearizing approximation used is rigorous only in the fluid core outside the boundary layer.

Table 1. Comparison of Results for the Entry Problem with Those of Others for Newtonian Fluids

(99% criterion)

	x/aN_{Re}	C
This work	0.122	1.33
Bogue	0.0575	1.16
Goldstein	—	1.41
Langhaar	0.115	1.28
Nikuradse (experimental)	0.125	1.32
Schiller	0.0575	1.16
Tomita	0.101	1.22

Nikuradse's results are experimental. Detailed description of the work is lacking.

The results presented in Table 1 indicate that the present analysis compares favorably with previous studies with the exception of those by Bogue and Schiller. Bogue and Schiller used an approximate momentum integral method of solution, the only difference being that Bogue included the radial momentum term neglected by Schiller. The momentum integral method assumes that boundary-layer theory is applicable throughout the entry region, which is inconsistent with the assumptions inherent in the theory. Hence the absolute magnitude of these results is in some doubt. Also Bogue has noted that his method is not valid for $n < 0.5$ because of inability to match the velocity profile in the boundary layer with a simple polynomial.

Tomita used a variational principle in solving the boundary-layer equations. This method requires assumption of a definite form for the equation from which the velocity profile is calculated and is more restrictive than the present analysis. Tomita's entry length results are compared with those of the present work in Figure 3, which indicates that the trend is the same, though absolute magnitudes compare only within 20%.

The pressure loss correction values presented by Tomita are erroneous, as the total correction is less than that obtained when only the kinetic energy increase of the fluid is considered. In all probability the error is a numerical one, as Tomita's equations were resolved yielding the results presented in Figure 7. Once again it is noted that the trend is the same as that found in the present work, but absolute magnitudes compare favorably only for $n < 0.4$.

Figures 8 and 9 compare calculated pressure losses in the entry region with experimental data. Figure 8 presents Newtonian data measured by Kreith and Eisenstadt (*7*), and Figure 9 presents carbopol and CMC data taken by Dodge (*3*). In comparing the theory with data of Dodge it was necessary to assume for each run that measurements from the first pressure tap, which was located slightly downstream of the entrance, were in agreement with

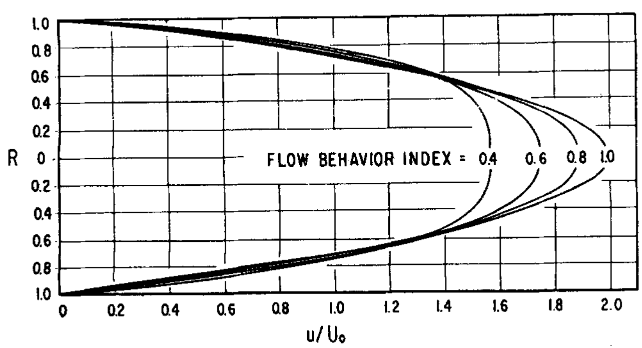

Fig. 6. Velocity profiles, $x = x_D$.

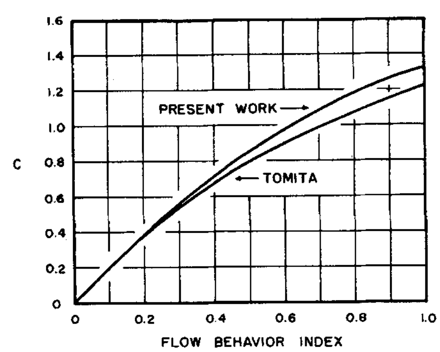

Fig. 7. Comparison with Tomita's pressure drop results.

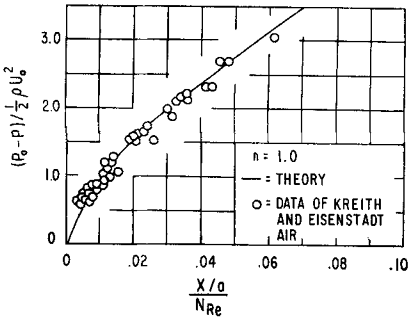

Fig. 8. Comparison of computed pressure loss with experimental data, $n = 1.0$.

theory. Then subsequent points for each run were located with respect to results obtained from the first pressure tap. Points marked with a "1" in Figure 9 indicate data obtained from the first pressure tap. Excellent agreement with the experimental results is obtained. It should be noted that Bogue, using a similar procedure, also obtained favorable agreement with his results.

It is apparent from Table 1 that the results of the present investigation agree more favorably with the experimental values for Newtonian fluids than do the results of others. This close agreement may be fortuitous, as experimental values are difficult to determine accurately.

Since Equation (1) makes no provision for elastic effects in the fluid, one would not expect the foregoing analysis to apply to highly elastic materials. Philippoff and Gaskins (*11*) have reported a number of experiments which seem to indicate an extremely short entry length for several viscoelastic fluids. It is of interest to note that approximate calculations, from Figure 2, predict $(x_D/a) << 1$ for the fluids and conditions reported by Philippoff and Gaskins.

ACKNOWLEDGMENT

The authors are grateful to the Research Corporation for partial support of this work. One of the authors gratefully acknowledges a fellowship from the E. I. DuPont de Nemours Company, Inc.

NOTATION

a = pipe radius
$A_1, A_2, \ldots A_5$ = constants defined by Equation (9)
c_i = constants defined by Equation (19)
$f_1, f_2, \ldots f_4$ = functions defined by Equations (8) and (A1) through (A3)

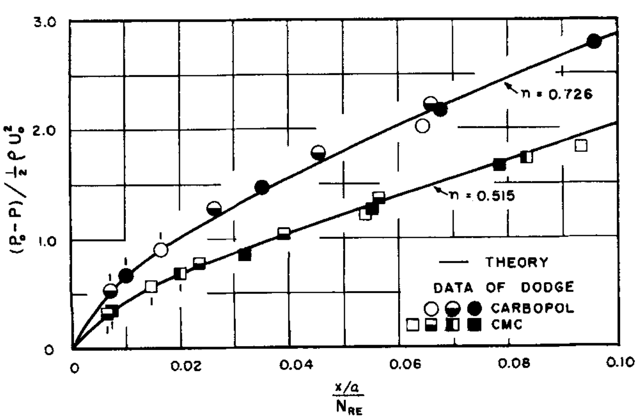

Fig. 9. Comparison of computed pressure loss with experimental data, $n < 1.0$.

I_2 = second invariant of rate of deformation tensor
K = consistency index defined by Equation (1)
K_i = constants defined by Equations (12)
n = flow behavior index
N_{Re} = Reynolds number, $\dfrac{(2a)^n U_o^{2-n} \rho}{K}$
p = pressure
R = dimensionless radial position, r/a
u = x component of velocity
u^* = perturbation velocity defined by Equation (19)
u_d^* = difference between u calculated from downstream analysis at the joining point and fully developed velocity
U = velocity outside boundary layer
U_o = average velocity
v = r component of velocity
X = $2^{n-1}\left(\dfrac{1+3n}{n}\right)^{n-2} (n+1)\,\sigma^{n+1}$
x = distance component in direction of fully developed flow

Greek Letters

η = $\left(1 - \dfrac{r^2}{a^2}\right)/2\sigma$
η_1 = value of η outside boundary layer
θ_i = function defined by Equation (22)
λ_i = eigenvalue defined by Equation (21)
μ_{eff} = effective viscosity
ρ = density
σ = $\left[\dfrac{2x}{aN_{Re}}\right]^{\frac{1}{n+1}}$
τ_{ij} = component of shear stress tensor
φ_i = function defined by Equation (21)
ψ = stream function defined by Equation (4)

Subscripts

D = evaluated at value of x where center-line velocity is 98% of fully developed value
j = evaluated at position where upstream and downstream solutions were joined

LITERATURE CITED

1. Acrivos, Andreas, M. J. Shah, and E. E. Petersen, *A.I.Ch.E. Journal*, **6**, 312 (1960).
2. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," pp. 101-104, Wiley, New York (1960).
3. Bogue, D. C., *Ind. Eng. Chem.*, **51**, 874 (1959).
4. Collins, Morton, Ph.D. thesis, Princeton Univ., Princeton, New Jersey (1962).
5. Collins, M., and W. R. Schowalter, *A.I.Ch.E. Journal*, **9**, 98 (1963).
6. Goldstein, S., "Modern Developments in Fluid Dynamics," Vol. 1, pp. 299-308, Oxford Press, London, England (1938).
7. Kreith, F., and R. Eisenstadt, *Trans. Am. Soc. Mech. Engrs.*, **79**, 1070 (1957).
8. Langhaar, H. L., *J. Appl. Mech.*, **9**, A55 (1942).
9. Nikuradse, J., in "Applied Hydro- and Aerodynamics," Prandtl and Tietjens, English Trans., p. 27, McGraw-Hill, New York (1950).
10. Oldroyd, J. G., in "Rheology," F. R. Eirich, ed., Vol. 1, pp. 653-682, Academic Press, New York (1956).
11. Philippoff, W., and F. H. Gaskins, *Trans. Soc. Rheol.*, **2**, 263 (1958).
12. Romanelli, M. J., in "Mathematical Methods for Digital Computers," A. Ralston and H. S. Wilf, ed., pp. 110-120, Wiley, New York (1960).
13. Schiller, L. *Zeit. für ang. Math. und Mech.*, **2**, 96 (1922).
14. Schowalter, W. R., *A.I.Ch.E. Journal*, **6**, 24 (1960).
15. Tomita, Y., *Bull. Japan Soc. Mech. Engrs.*, **4**, 77 (1961).

Manuscript received October 8, 1962; revision received May 24, 1963; paper accepted June 7, 1963.